MATH 141 HOMEWORK 5 DUE DATE: TUESDAY, MAY 30TH

Name: Solutions

This homework is due Tuesday, May 30th during recitation. If you have questions regarding any of this, feel free to ask during office hours or send me an email. When writing solutions, present your answers clearly and neatly, showing only necessary work.

1. Find the value or values of c that satisfy the equation $\frac{f(b)-f(a)}{b-a} = f'(c)$ in the conclusion of the Mean Value Theorem for the following functions in the given interval.

(a) $f(x) = x^{2/3}$, $[0, 1]$

$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0}$

$= \frac{1-0}{1-0} = 1$

$f'(x) = \frac{2}{3}x^{-1/3} = 1$

$\Rightarrow x^{1/3} = 2/3$

$\Rightarrow x = 8/27$

Answer: $c = 8/27$

(b) $f(x) = \sin^{-1}(x)$, $[-1, 1]$

$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)}$

$= \frac{\pi/2-(-\pi/2)}{2} = \pi/2$

$f'(x) = \frac{1}{\sqrt{1-x^2}} = \pi/2$

$\Rightarrow \sqrt{1-x^2} = \frac{2}{\pi}$

$\Rightarrow x^2 = 1 - 4/\pi^2$

Answer: $c = \pm\sqrt{1-4/\pi^2}$

(c) $f(x) = x^3 - x^2$, $[-1, 2]$

$\frac{f(b)-f(a)}{b-a} = \frac{f(2)-f(-1)}{2-(-1)}$

$= \frac{4-(-2)}{3} = 2$

$f'(x) = 3x^2 - 2x = 2$

$\Rightarrow 3x^2 - 2x - 2 = 0$

$x = \frac{-(-2)\pm\sqrt{(-2)^2-4(3)(-2)}}{2(3)}$

Answer: $c = \frac{1}{3}(1\pm\sqrt{7})$

2. Show that the function $f(x) = x^7 + x^5 + x^3 + 1$ has exactly 1 real root.

$f(-1) = (-1)^7 + (-1)^5 + (-1)^3 + 1$

$= -2$

$f(1) = 1^7 + 1^5 + 1^3 + 1$

$= 4$

Sign change $\xrightarrow{\text{IVT}}$ root, x_0, in the interval $[-1, 1]$.

Suppose there exists $x_1 \neq x_0$ such that $f(x_1) = 0$. Then by Rolles Theorem there exists a point c with $f'(c) = 0$ and $f'(x) < 0$ in either $(c-\varepsilon, c)$ or $(c, c+\varepsilon)$

$f'(x) = 7x^6 + 5x^4 + 3x^2 \geq 0$

for all x. Thus no such point exists and so $f(x)$ has exactly one root.

3. Evaluate $\lim_{x\to 0} \frac{2\sin(5x)}{3x}$

$$\lim_{x\to 0} \frac{2\sin(5x)}{3x} \overset{L'H}{=\!=} \lim_{x\to 0} \frac{10\cos(5x)}{3} = \frac{10\cdot 1}{3}$$

Answer: $10/3$

4. Evaluate $\lim_{x\to\infty} \left(1+\frac{b}{x}\right)^{kx}$ (where b and k are arbitrary constants)

$$\ln\left(\lim_{x\to\infty}\left(1+\frac{b}{x}\right)^{kx}\right) = \lim_{x\to\infty} \ln\left[\left(1+b/x\right)^{kx}\right]$$

$$= \lim_{x\to\infty} kx\ln(1+b/x) = k\lim_{x\to\infty} \frac{\ln(1+b/x)}{1/x}$$

$$\overset{L'H}{=\!=} k\lim_{x\to\infty} \frac{\frac{1}{1+b/x}\left(\frac{-b}{x^2}\right)}{-1/x^2} = k\lim_{x\to\infty} \frac{b}{1+b/x} = kb$$

$$\Longrightarrow \lim_{x\to\infty}\left(1+b/x\right)^{kx} = e^{kb}$$

Answer: e^{kb}

Cont.

5. A spherical iron ball 8in in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10\text{in}^3/\text{min}$;

(a) How fast is the thickness of the ice decreasing when it is 2in thick?

$$V = \frac{4}{3}\pi r^3 - \frac{4}{3}\pi 4^3$$

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt}$$

$$\Longrightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt} = \frac{1}{4\pi(4+2)^2}(10) = \frac{5}{72\pi}$$

Answer: $\frac{5}{72\pi}$ in/sec

(b) How fast is the outer surface area of ice decreasing?

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = 4\pi(2r)\frac{dr}{dt}$$

$$= 4\pi(2(4+2)) \cdot \frac{5}{72\pi} = \frac{10}{3}$$

Answer: $\frac{10}{3}$ in^2/sec

The End.